CARINGBAH HIGH

Student Name/Number: ______________________

Class: ______________________

2023 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics Advanced

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen.
- Calculators approved by NESA may be used.
- Reference sheet is provided.
- For questions in Section II, show relevant mathematical reasoning and/or calculations.

Total marks: 100

Section I – 10 marks (pages 3 - 6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section.
- Answer the questions on page 35.

Section II – 90 marks (pages 8 – 29)

- Attempt Questions 11–36
- Allow about 2 hours and 45 minutes for this section.

Marker's Use Only							
Section I	**Section II**						**Total**
Q1-10	**11 – 16**	**17 – 21**	**22 – 25**	**26– 30**	**31- 33**	**34- 36**	
/10	/18	/16	/14	/17	/14	/11	/100

Section I

10 marks
Attempt Question 1-10
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1-10.

---

1 What are the solutions to the equation $2\cos x = \sqrt{3}$, where $0 \leq x \leq 2\pi$?

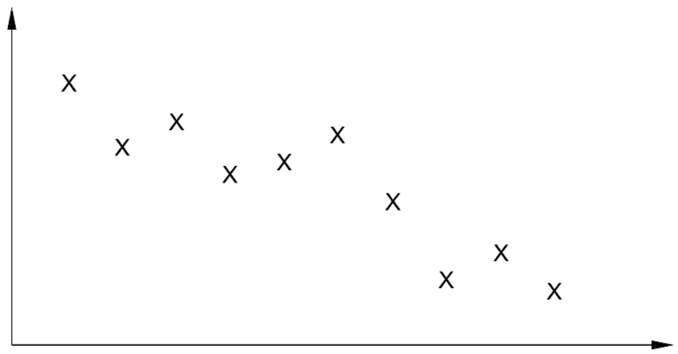
(A) $\frac{\pi}{3}$ and $\frac{5\pi}{3}$

(B) $\frac{\pi}{3}$ and $\frac{2\pi}{3}$

(C) $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

(D) $\frac{\pi}{6}$ and $\frac{11\pi}{6}$

2 Consider the bivariate data shown on the scatterplot below.



Which of the following values is the best estimate for Pearson's correlation coefficient for this data?

(A) -0.9

(B) -0.2

(C) 0.9

(D) 0.2

3 What is the derivative of e^{x^6}?

(A) $6x^5e^{x^6}$

(B) $6xe^{x^6}$

(C) $6x^5e^{6x^5}$

(D) $x^6e^{x^6-1}$

4 An infinite geometric series has a first term of 10 and a limiting sum of 30. What is the common ratio?

(A) $\frac{1}{3}$

(B) $\frac{1}{2}$

(C) $\frac{2}{3}$

(D) $\frac{3}{4}$

5 The inequality which defines the domain of the function $f(x) = \frac{-4}{\sqrt{9-x^2}}$ is:

(A) $x \leq 3$

(B) $-3 < x < 3$

(C) $-3 \leq x < 3$

(D) $x < -3, x > 3$

6 Given that $\tan\theta = \frac{3}{2}$ for $0 < \theta < \pi$, what is the exact value of $\sin\theta$?

(A) $\frac{-2}{\sqrt{13}}$

(B) $\frac{3}{\sqrt{13}}$

(C) $\frac{-3}{\sqrt{13}}$

(D) $\frac{2}{\sqrt{13}}$

7 A function is given by the rule $f(x) = \begin{cases} (x-2)^2, & x \le 2 \\ 4x-7, & x > 2 \end{cases}$.
What is the value of $f\big(f(0)\big)$?

(A) 5

(B) -7

(C) 9

(D) 4

8 What is the period for the function $f(x) = -3\sin\left(\frac{\pi x}{5}\right)$?

(A) 5

(B) 5π

(C) 10

(D) 10π

9 A and B are two chance events such that $P(A) = 0.45$, $P(B) = 0.3$ and $P(A|B) = 0.6$.
What is the value of $P(B|A)$?

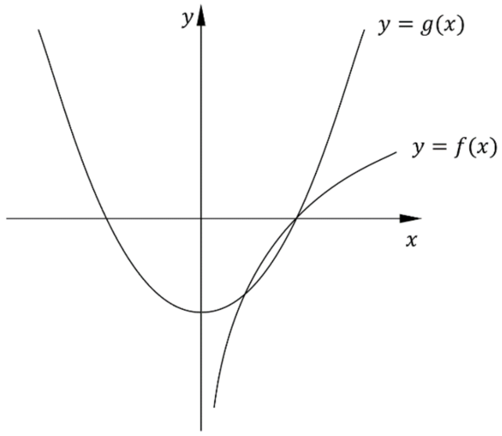
(A) 0.4

(B) 0.5

(C) 0.8

(D) 0.9

10 The graph shows $y = f(x)$ and $y = g(x)$, where $f(x) = \ln x$ and $g(x) = x^2 - 1$.



How many solutions does the equation $[f(x)]^2 - [g(x)]^2 = 0$ have?

(A) 0

(B) 1

(C) 2

(D) 3

END OF SECTION 1

Marks

Question 11

Differentiate y with respect to x:

a) **2**

$$y = \frac{3x + 1}{x + 4}$$

b) **2**

$$y = x\sqrt{4x^2 - 1}$$

Question 12 **2**

Solve $|5 - 4x| = 11$

Question 13 **3**

Evaluate the expression below, expressing your answer as an integer:

$$\int_{\ln 2}^{2\ln 2} e^{2x}\,dx$$

Question 14

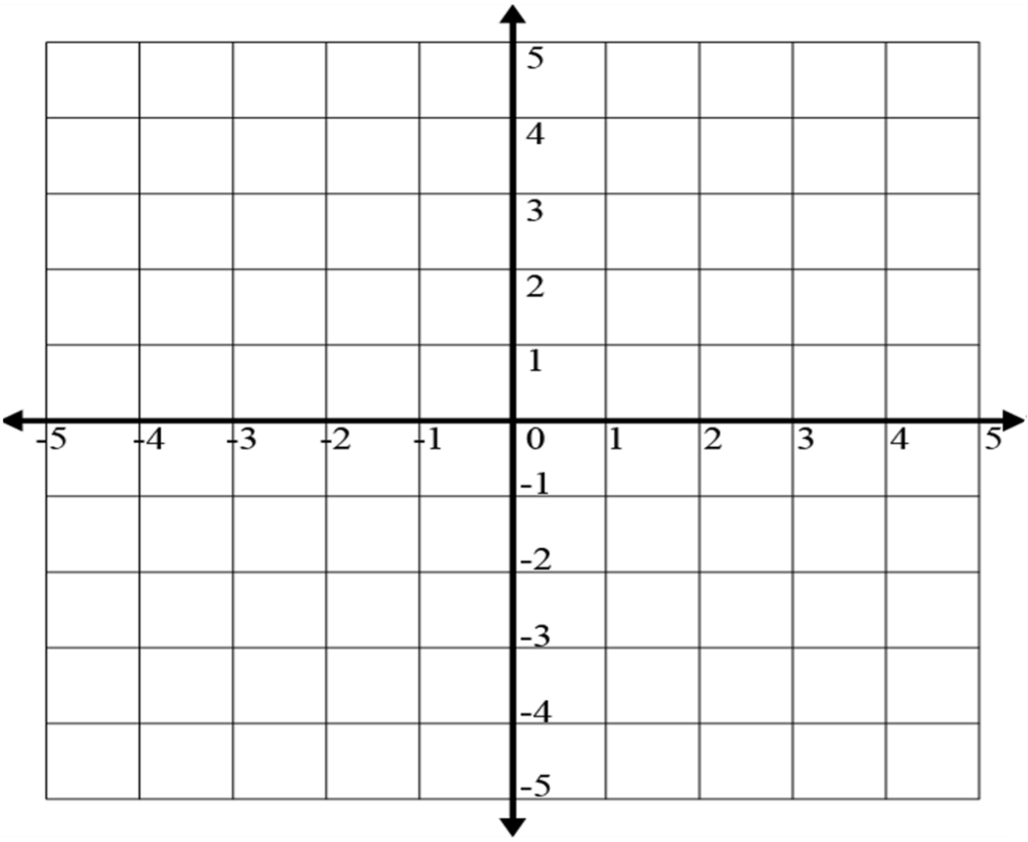
It is given that $f''(x) = 6x$ and that $f(x)$ has a stationary point at $(-1,2)$. Find $f(x)$. **3**

Question 15 **3**

Find the equation of the normal to the curve $y = 2(5x-4)^4$ at $x = 1$.

Question 16 **3**

Sketch $y = \ln(x+4) - 2$ showing **all** the key features and clearly labelling any intercepts **in exact form**.



Question 17

It is given that $1 + 2x - 3x^2 \geq 0$ in the domain $[0,1]$.

a) Prove that $f(x) = 1 + 2x - 3x^2$ is a probability density function for $[0,1]$. **2**

b) State the mode of the probability density function. **1**

Question 18 **3**

The circle $x^2 + y^2 - 6x + 8y - 11 = 0$ is transformed by a horizontal translation to the left by 4 units and a vertical translation up 3 units.
What is the centre and radius of the new circle?

Question 19

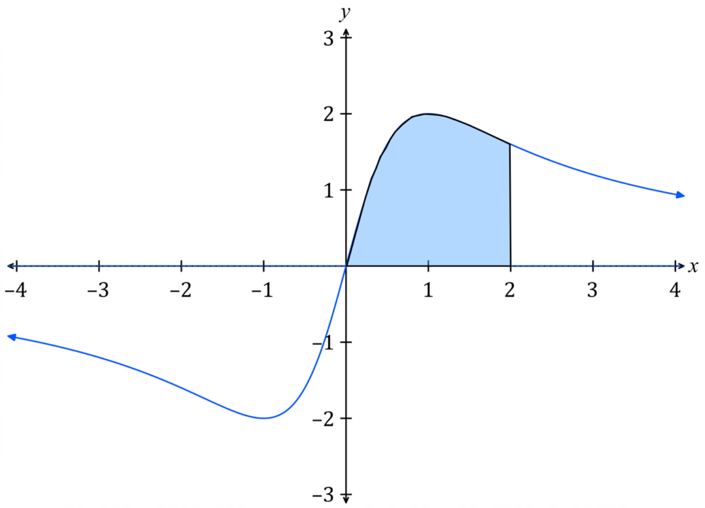
a) Find $\int \sin 3x \, dx$ **1**

2

b) Evaluate $\int_0^1 2x\,(x^2 + 2)^3 \, dx$

Question 20 **3**

The diagram below shows the graph of $y = \frac{4x}{x^2+1}$



The region enclosed by the graph, the x-axis, and the line $x = 2$ is shaded.
Calculate the exact value of the area of the shaded region.

Continue to next page.

Question 21

a) If $y = \frac{x^2+2x}{(x+1)^2}$. Show that $\frac{dy}{dx} = \frac{2}{(x+1)^3}$. **3**

b) Find the set of values of x for which the function $y = \frac{x^2+2x}{(x+1)^2}$ is increasing. **1**

Continue to next page.

Question 22

a) Show that **2**

$$\frac{\cos^2\theta}{1-\sin\theta}-\frac{\cos^2\theta}{1+\sin\theta}=2\sin\theta$$

b) Hence, solve: **2**

$$\frac{\cos^2\theta}{1-\sin\theta}-\frac{\cos^2\theta}{1+\sin\theta}=1 \quad \text{for } 0\le\theta\le\pi\,.$$

APPROXIMATELY HALFWAY – 48 marks out of 100 complete at this point

Question 23

The population, P, of parrots, is decreasing at a rate proportional to P:

$$\frac{dP}{dt} = -kP$$

Initially in August 2003 there were 3000 parrots, and by August 2013 the population had decreased to 2750. Note that t is in years.

a) Show that $P = 3000e^{-kt}$ is a solution to the above differential equation. **1**

b) Find the value the k (Answer correct to four decimal places). **2**

c) If the population continues to decrease at this rate what will be the expected population in August 2023? (Answer to the nearest whole number). **1**

Question 24

2

Use the trapezoidal rule with four sub-intervals to evaluate the integral below.
Answer in simplified fraction form.

$$A = \int_1^5 \frac{10}{x+1}\,dx$$

Continue to next page.

Question 25

The random variable X has this probability distribution.

X	11	12	13	14	15
$P(X = x)$	0.3	0.2	0.1	0.3	0.1

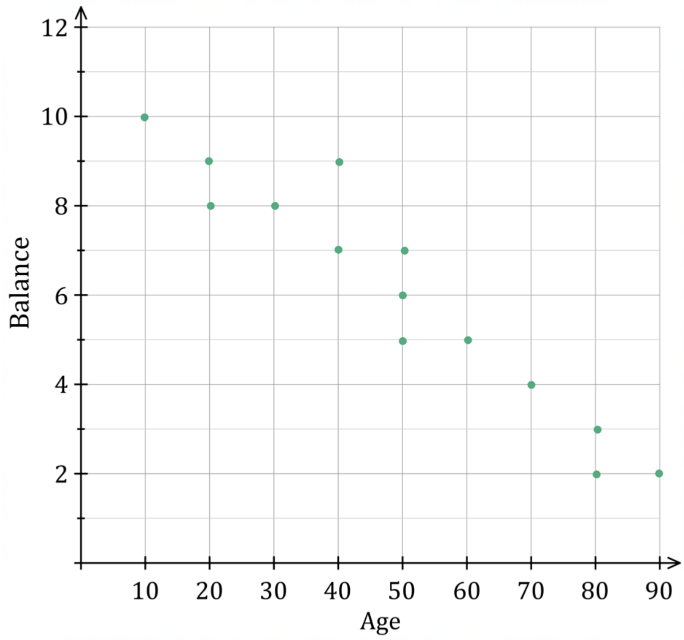
a) Find the expected value. **1**

b) Find the standard deviation of x correct to 2 decimal places. **3**

Continue to next page.

Question 26

The scatterplot below shows the relationship between age and balance.



a) Draw a line of best fit on the scatterplot **and** find the equation of this line. **3**

b) Hannah is 42 years old. Use your equation to find what is her expected balance? **1**

c) Use the table of values below to find the value of the Pearson's correlation coefficient correct to two decimal places. **1**

Age	10	20	20	30	40	40	50	50	50	60	70	80	80	90
Balance	10	8	9	8	7	9	5	6	7	5	4	2	3	2

Question 27

The first 3 terms of a sequence are:

$$(x-1), (3x+2), (5x+5), \ldots\ldots\ldots$$

a) Show that the sequence is arithmetic. **1**

b) Find the sum of the first 50 terms of the sequence, in terms of x. **2**

Question 28 **2**

The table below gives some values of the probabilities of a standard normal distribution.

z	first decimal place									
	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0.	0.5000	0.5398	0.5793	0.6179	0.6554	0.6915	0.7257	0.7580	0.7881	0.8159
1.	0.8413	0.8643	0.8849	0.9032	0.9192	0.9332	0.9452	0.9554	0.9641	0.9713
2.	0.9772	0.9821	0.9861	0.9893	0.9918	0.9938	0.9953	0.9965	0.9974	0.9981
3.	0.9987	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

Use the table above to find the probability $P(-1.5 \leq Z < 2.2)$

Question 29

The particle's displacement is given by $x = (t^2 + 1)e^{-t}$ metres and velocity v m/s.

a) Find the initial displacement of the particle. **1**

b) Express the velocity v in factorised form. **2**

c) Hence, find when the particle is at rest. **1**

Continue to next page.

Question 30 **3**

Solve $\ln(2\sin^2\theta\ -\cos\theta) = 0$ for $0 \leq \theta \leq 2\pi$.

Question 31 **3**

Evaluate

$$\int_0^{\frac{\pi}{4}} (\sin^2 x + \cos^2 x + \tan^2 x)\, dx$$

APPROXIMATELY THREE QUARTERS COMPLETE – 78 marks out of 100 complete

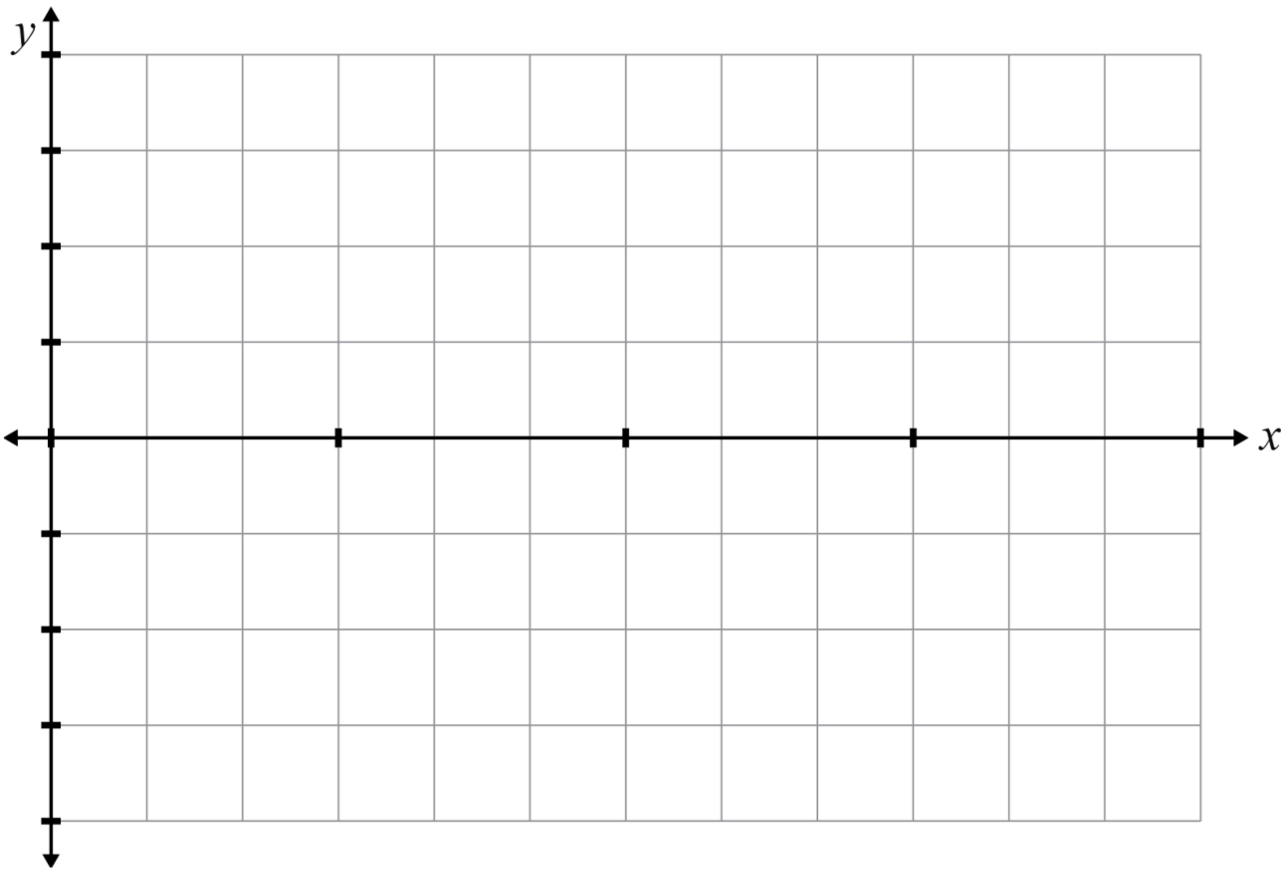
Question 32

The function $f(x) = \cos(3x) - 1$ is defined in the interval $0 \leq x \leq \frac{2\pi}{3}$.

a) What is the amplitude and period of the function? **2**

b) What is the range of the function? **1**

c) Sketch $f(x) = \cos(3x) - 1$ in the interval $0 \leq x \leq \frac{2\pi}{3}$. **2**



Continue to next page.

Question 33

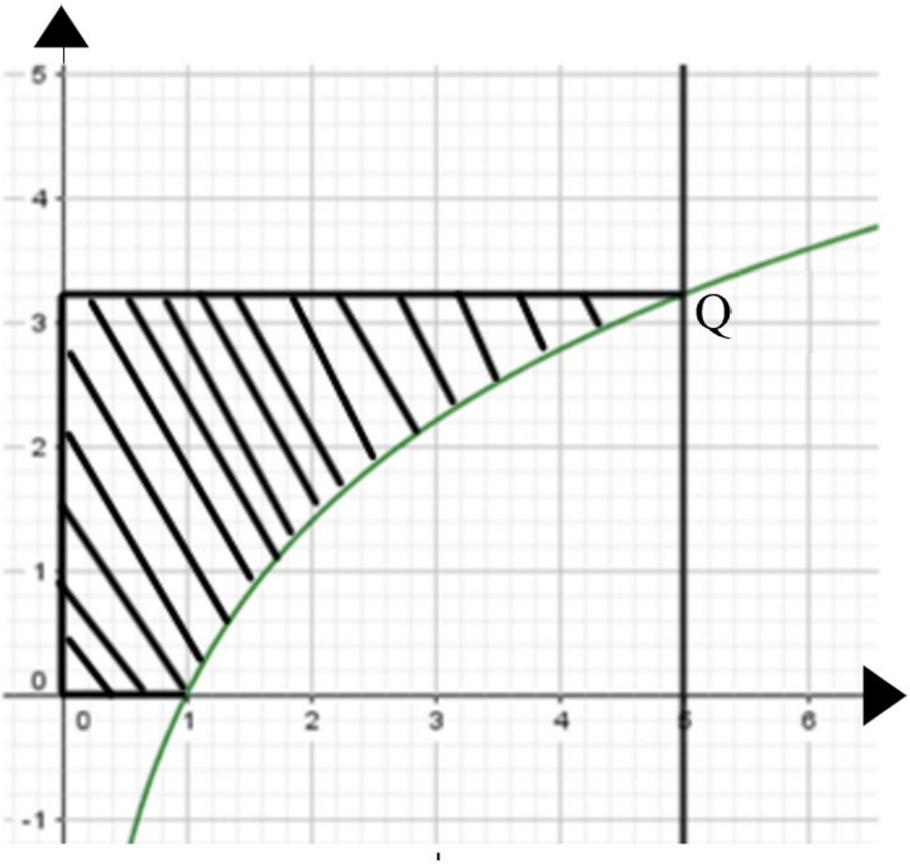
a) Show that **2**

$$\frac{d}{dx}(x\log_{e}(x) - x) = \log_{e} x$$

b) Hence or otherwise find $\int 2\log_{e}(x)\,dx$ **1**

Continue to next page.

a) The graph shows the curve $y = 2\log_e(x)$ which meets the line $x = 5$ at Q.



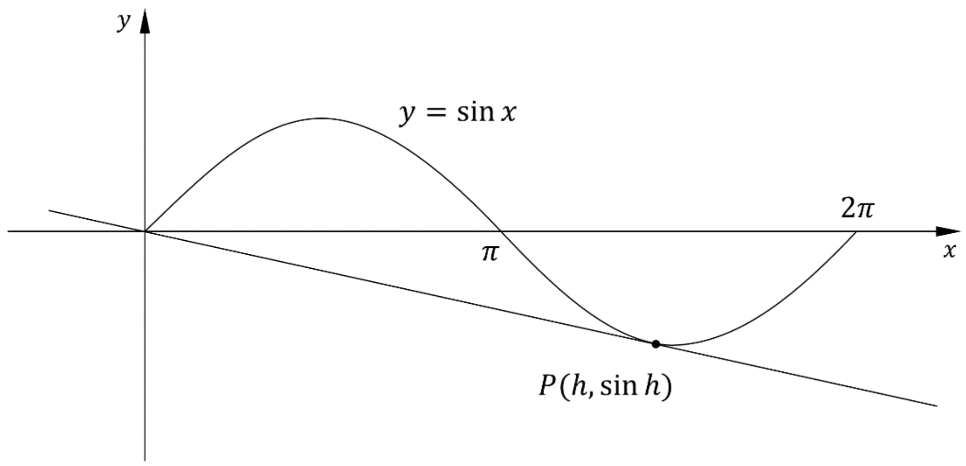
Using your answers from (i) and (ii), or otherwise, find the exact area of the shaded section.

Question 34 **3**

The graph $y = \sin x$ only has one tangent in the domain $[\pi, 2\pi]$ that passes through the origin.

Let the point of contact of this tangent be $P(h, \sin h)$.

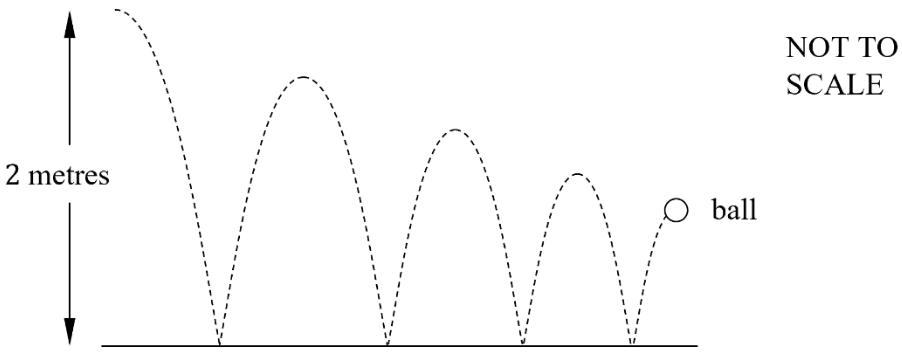
Prove that $h = \tan h$.



Question 35 **3**

The Moon has a lower gravity than Earth, and there is no atmosphere to cause air resistance, so a ball would bounce higher and for much longer on the Moon than on Earth.

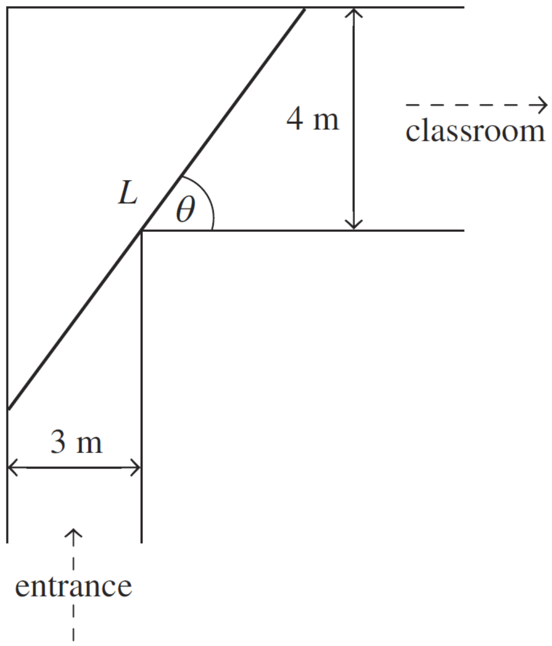
When a ball is dropped on the Moon each bounce is 95% as high as the previous bounce. When an identical ball is dropped on Earth each bounce is 50% as high as the previous bounce.



Two identical balls are dropped on the Moon and on Earth, each from a height of two metres. Calculate the difference in the total vertical distance travelled by these balls.

Question 36

A new whiteboard is being moved into a classroom. The whiteboard must be taken from the entrance, through the school's corridors and into the classroom. Two of the corridors are perpendicular to each other. The first corridor is 3 metres wide and the second corridor is 4 metres wide, as shown in the diagram. The length of the whiteboard is shown using L.



The whiteboard makes an angle θ to the horizontal on the corner of the corridors such that $0 < \theta < 90$.

a) Show that the length of the whiteboard, L, is **2**

$$\frac{3}{\cos\theta} + \frac{4}{\sin\theta}$$

Continue to next page.

b) In order to find the maximum possible length of the whiteboard such that it can be carried around the corner, you **must find the angle θ that minimises the function L** and then use that angle to **find that length of the whiteboard**. Hence, find that value of theta to the nearest minute and the length L to the nearest metre. **3**

END OF EXAM

Solutions

Student Name/Number: ____________________

Class: ____________________

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2023 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics Advanced

General Instructions

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/10	/18	/16	/14	/17	/14	/11	/100

Section I

10 marks
Attempt Question 1-10
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1-10.

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1 What are the solutions to the equation $2\cos x = \sqrt{3}$, where $0 \le x \le 2\pi$?

(A) $\frac{\pi}{3}$ and $\frac{5\pi}{3}$

(B) $\frac{\pi}{3}$ and $\frac{2\pi}{3}$

(C) $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

(D) $\frac{\pi}{6}$ and $\frac{11\pi}{6}$

$\cos x = \frac{\sqrt{3}}{2}$

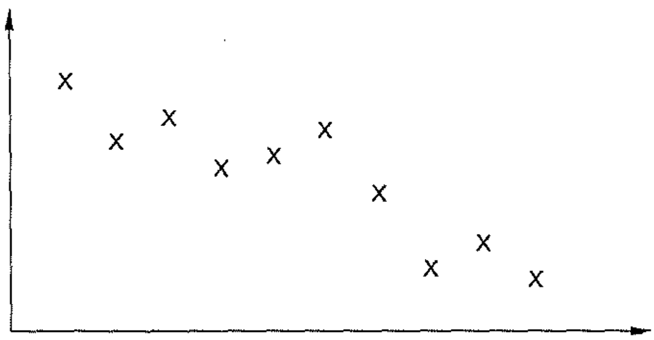
S A
T C

$\frac{\pi}{6}$

$2\pi - \frac{\pi}{6}$

$2\pi - \frac{\pi}{6}$
$= \frac{12\pi - \pi}{6}$
$= \frac{11\pi}{6}$

2 Consider the bivariate data shown on the scatterplot below.



Which of the following values is the best estimate for Pearson's correlation coefficient for this data?

(A) -0.9

(B) -0.2

(C) 0.9

(D) 0.2

3 What is the derivative of e^{x^6}?

(A) $6x^5e^{x^6}$

(B) $6xe^{x^6}$

(C) $6x^5e^{6x^5}$

(D) $x^6e^{x^6-1}$

$f(x)=e^{x^6}$
$f'(x)=6x^5e^{x^6}$

4 An infinite geometric series has a first term of 10 and a limiting sum of 30. What is the common ratio?

(A) $\frac{1}{3}$

(B) $\frac{1}{2}$

(C) $\frac{2}{3}$

(D) $\frac{3}{4}$

$S_\infty=\frac{a}{1-r}$
$30=\frac{10}{1-r}$
$30-30r=10$
$30r=20 \Rightarrow r=2/3$

5 The inequality which defines the domain of the function $f(x)=\frac{-4}{\sqrt{9-x^2}}$ is:

(A) $x \leq 3$

(B) $-3 < x < 3$

(C) $-3 \leq x < 3$

(D) $x < -3, x > 3$

$9-x^2>0$
$x^2<9$
$-3<x<3$

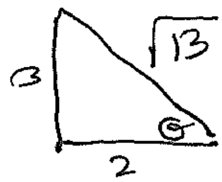
6 Given that $\tan\theta = \frac{3}{2}$ for $0 < \theta < \pi$, what is the exact value of $\sin\theta$?

(A) $\frac{-2}{\sqrt{13}}$

(B) $\frac{3}{\sqrt{13}}$

(C) $\frac{-3}{\sqrt{13}}$

(D) $\frac{2}{\sqrt{13}}$



$\tan\theta>0$

$\sin\theta=\frac{3}{\sqrt{13}}$

7 A function is given by the rule $f(x) = \begin{cases} (x-2)^2, & x \le 2 \\ 4x-7, & x > 2 \end{cases}$.

What is the value of $f(f(0))$?

$f(0) = (0-2)^2 = 4$

$f(4) = 16-7$
$= 9$

(A) 5

(B) −7

(C) 9

(D) 4

8 What is the period for the function $f(x) = -3\sin\left(\frac{\pi x}{5}\right)$?

(A) 5

(B) 5π

(C) 10

(D) 10π

$2\pi \div \frac{\pi}{5}$

$2\cancel{\pi} \times \frac{5}{\cancel{\pi}} = 10$

9 A and B are two chance events such that $P(A) = 0.45$, $P(B) = 0.3$ and $P(A|B) = 0.6$.
What is the value of $P(B|A)$?

(A) 0.4

(B) 0.5

(C) 0.8

(D) 0.9

$P(B|A) = \frac{P(A \cap B)}{P(A)}$

$= \frac{0.18}{0.45}$

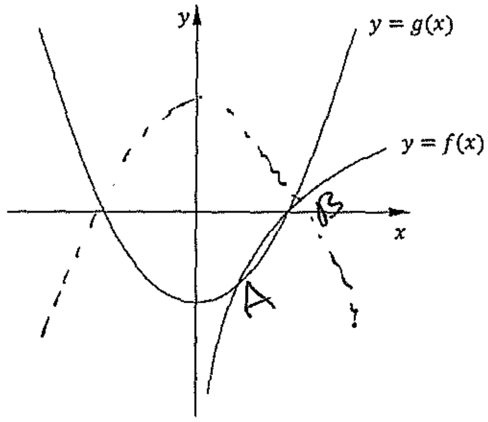
$= 0.4$

$P(A \cap B) = P(A|B) \times P($

$= 0.6 \times 0.3$

$= 0.18$

10 The graph shows $y = f(x)$ and $y = g(x)$, where $f(x) = \ln x$ and $g(x) = x^2 - 1$.



How many solutions does the equation $[f(x)]^2 - [g(x)]^2 = 0$ have?

(A) 0

(B) 1

(C) 2

(D) 3

$(f(x) - g(x))(f(x) + g(x)) = 0$

$f(x) = g(x)$ or $f(x) = -g(x)$

$\Downarrow$ 2 : A & B

$\Downarrow$ red sketch 1 : B

2 solutions

END OF SECTION 1

Marks

Question 11

Differentiate y with respect to x:

a) **2**

$$y = \frac{3x+1}{x+4}$$

$u = 3x+1$ $\quad$ $v = x+4$

$u' = 3$ $\quad$ $v' = 1$

$$y' = \frac{3(x+4) - (3x+1)}{(x+4)^2} \quad — \quad 1$$

$$y' = \frac{11}{(x+4)^2} \quad \text{or} \quad \frac{11}{x^2+8x+16} \quad — \quad 1$$

b) **2**

$$y = x\sqrt{4x^2-1}$$

$u = x$ $\quad$ $v = (4x^2-1)^{\frac{1}{2}}$

$u' = 1$ $\quad$ $v' = \frac{1}{2}(4x^2-1)^{-\frac{1}{2}} \times 8x$ $\quad$ } — 1

$$y' = \sqrt{4x^2-1} \times 1 - \frac{4x^2}{\sqrt{4x^2-1}}$$

$$= \frac{4x^2-1+4x^2}{\sqrt{4x^2-1}}$$

or with indices

1

$$= \frac{8x^2-1}{\sqrt{4x^2-1}}$$

Question 12 **2**

Solve $|5 - 4x| = 11$

$5 - 4x = 11$ or $5 - 4x = -11$

$-4x = -6$ or $-4x = -16$ — Correct working out — 1

$x = -\frac{3}{2}$ or $x = 4$ — 1

Question 13 **3**

Evaluate the expression below, expressing your answer as an integer:

$$\int_{\ln 2}^{2\ln 2} e^{2x}\,dx$$

$= \frac{1}{2}\left[e^{2x}\right]_{\ln 2}^{2\ln 2}$ — 1

$= \frac{1}{2}\left[e^{4\ln 2} - e^{2\ln 2}\right]$ — 1

$= \frac{1}{2}\left[2^4 - 2^2\right]$

$= \frac{1}{2}(12)$

$= 6$ — 1

Question 14

It is given that $f''(x) = 6x$ and that $f(x)$ has a stationary point at $(-1,2)$. Find $f(x)$. **3**

$f'(x) = \int 6x \, dx$

$f'(x) = 3x^2 + C$

$* f'(1) = 0$

$3 + C = 0$

$\boxed{C = -3}$ — 1

$* f(x) = \int 3x^2 - 3$

$= \frac{3x^3}{3} - 3x + C$ — 1

$* f(-1) = -1 + 3 + C = 2$

$\boxed{C = 0} \Longrightarrow \boxed{f(x) = x^3 - 3x}$ — 1

Question 15 **3**

Find the equation of the normal to the curve $y = 2(5x-4)^4$ at $x = 1$.

$y' = 8(5x-4)^3 \times 5 = 80(5x-4)^3$ — 1

$m_{Tang} = y'(x=1) = 40 \Rightarrow \boxed{m_{Normal} = -\frac{1}{40}}$ — 1

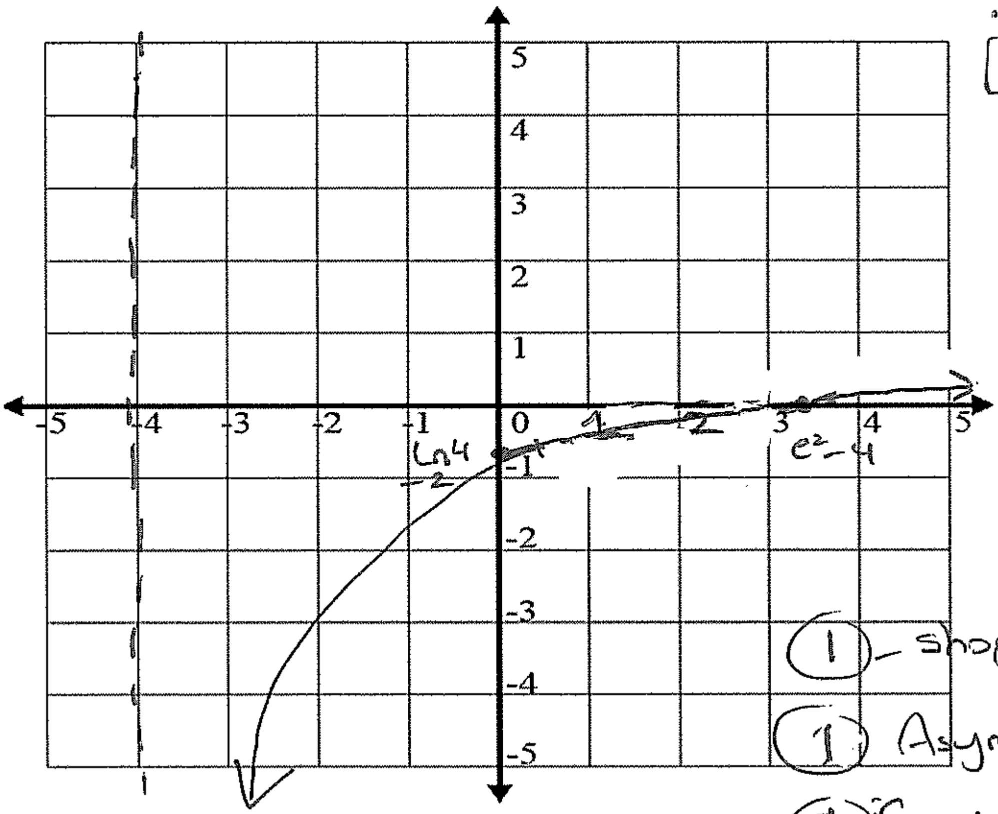
$y - 2 = -\frac{1}{40}(x-1)$

$\boxed{y = \frac{-1}{40} + \frac{81}{40}}$ — 1

or $2\frac{1}{40}$

Question 16 **3**

Sketch $y = \ln(x+4) - 2$ showing **all** the key features and clearly labelling any intercepts **in exact form**.



$\cdot\, x = 0$

$\boxed{y = \ln 4 - 2}$

$\simeq -0.614$

$\cdot\, y = 0$

$\ln(x+4) = 2$

$x + 4 = e^2$

$\boxed{x = e^2 - 4}$

~ 3.38

1 – shape of curve

1 Asymptotes

1 Exact intercepts

Question 17

It is given that $1 + 2x - 3x^2 \geq 0$ in the domain $[0,1]$.

2

a) Prove that $f(x) = 1 + 2x - 3x^2$ is a probability density function for $[0,1]$.

$$\int_0^1 1 + 2x - 3x^2 = \left[x + x^2 - x^3\right]_0^1 \quad - 1$$

$$= (1 + 1 - 1) - 0$$

$$= 1 \quad - 1$$

1

b) State the mode of the probability density function.

$$\frac{-b}{2a} = \frac{-2}{-6} = \boxed{\frac{1}{3} = x} \quad - 1$$

Question 18 3

The circle $x^2 + y^2 - 6x + 8y - 11 = 0$ is transformed by a horizontal translation to the left by 4 units and a vertical translation up 3 units.
What is the centre and radius of the new circle?

$x^2 - 6x + y^2 + 8y = 11$

$x^2 - 6x + 9 + y^2 + 8y + 16 = 11 + 9 + 16$

$(x-3)^2 + (y+4)^2 = 36$ — 1

centre $(3, -4)$

New centre $= (-1, -1)$ 1

radius $r = 6$ — 1

Question 19

a) Find $\int \sin 3x \, dx$ 1

$-\frac{1}{3}\cos 3x + C$ — 1

2

b) Evaluate $\int_0^1 2x\,(x^2+2)^3 \; dx$

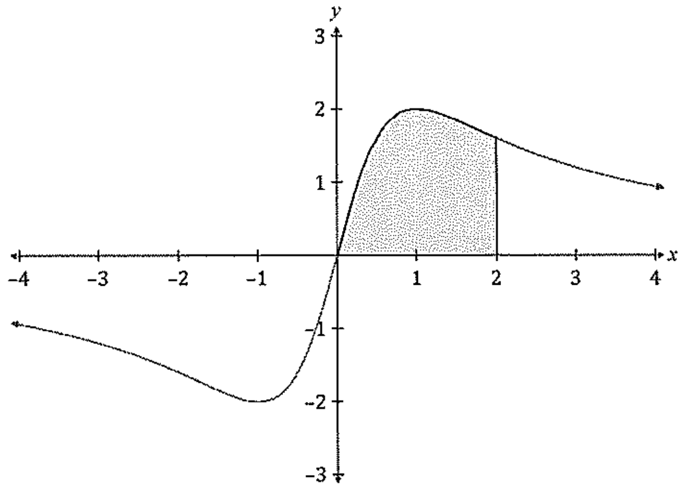
$\left[\frac{(x^2+2)^4}{4}\right]_0^1$ — 1

$= \frac{3^4}{4} - \frac{2^4}{4}$

$= \frac{65}{4}$ — 1

Question 20 **3**

The diagram below shows the graph of $y = \frac{4x}{x^2+1}$



The region enclosed by the graph, the x-axis, and the line $x = 2$ is shaded.
Calculate the exact value of the area of the shaded region.

$$A = \int_0^2 \frac{4x}{x^2+1}\,dx \quad — \quad 1$$

$$= 2\int_0^2 \frac{2x}{x^2+1}\,dx$$

$$= 2\left[\ln|x^2+1|\right]_0^2 \quad — \quad 1$$

$$= 2[\ln 5 - \ln 1] \quad 1$$

$$= 2\ln 5 \quad u^2$$

Continue to next page.

Question 21

a) If $y = \frac{x^2+2x}{(x+1)^2}$. Show that $\frac{dy}{dx} = \frac{2}{(x+1)^3}$. **3**

$u = x^2+2x \qquad v = (x+1)^2$

$u' = 2x+2 \qquad v' = 2(x+1)$ — 1

$= \frac{(2x+2)(x+1)^2 - 2(x+1)(x^2+2x)}{(x+1)^4}$ — 1

$= \frac{2(x+1)[x^2+2x+1-x^2-2x]}{(x+1)^4}$

$= \frac{2}{(x+1)^3}$ — 1

b) Find the set of values of x for which the function $y = \frac{x^2+2x}{(x+1)^2}$ is increasing. **1**

$y' > 0 \Rightarrow \boxed{x > -1}$ — 1

Continue to next page.

Question 22

a) Show that

$$\frac{\cos^2\theta}{1-\sin\theta} - \frac{\cos^2\theta}{1+\sin\theta} = 2\sin\theta$$ **2**

$$LHS = \frac{\cos^2\theta(1+\sin\theta) - \cos^2\theta(1-\sin\theta)}{1-\sin^2\theta}$$ ①

$$= \frac{\cancel{\cos^2\theta} + \cos^2\theta\sin\theta - \cancel{\cos^2\theta} + \cos^2\theta\sin\theta}{1-\sin^2\theta}$$

$$= \frac{2\cancel{\cos^2\theta}\sin\theta}{\cancel{\cos^2\theta}}$$

$$= 2\sin\theta = RHS$$ ①

b) Hence, solve:

$$\frac{\cos^2\theta}{1-\sin\theta} - \frac{\cos^2\theta}{1+\sin\theta} = 1 \quad \text{for } 0 \le \theta \le \pi.$$ **2**

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

① ①

APPROXIMATELY HALFWAY – 48 marks out of 100 complete at this point

Question 23

The population, P, of parrots, is decreasing at a rate proportional to P:

$$\frac{dP}{dt} = -kP$$

Initially in August 2003 there were 3000 parrots, and by August 2013 the population had decreased to 2750. Note that t is in years.

a) Show that $P = 3000e^{-kt}$ is a solution to the above differential equation. **1**

$$\frac{dP}{dt} = -3000ke^{-kt}$$

$$= -kP$$

b) Find the value the k (Answer correct to four decimal places). **2**

$$2750 = 3000e^{-10k}$$

$$e^{-10k} = \frac{2750}{3000} \quad — (1)$$

$$-10k = \ln\left(\frac{2750}{3000}\right)$$

$$k = \frac{-\ln\left(\frac{275}{300}\right)}{10}$$

$$\boxed{k \simeq 0.0087} \quad — (1)$$

c) If the population continues to decrease at this rate what will be the expected population in August 2023? (Answer to the nearest whole number). **1**

$t = 20$, $P = 3000e^{-0.0087(20)}$

$$\simeq \boxed{2521} \quad — (1)$$

Question 24 2

Use the trapezoidal rule with four sub-intervals to evaluate the integral below.
Answer in simplified fraction form.

$$A = \int_1^5 \frac{10}{x+1}\,dx$$

x	1	2	3	4	5
y	5	$\frac{10}{3}$	$\frac{5}{2}$	2	$\frac{5}{3}$

$$A = \frac{4}{8}\left[5 + \frac{5}{3} + 2\left(\frac{10}{3} + \frac{5}{2} + 2\right)\right]$$ — 1

$$A = \frac{1}{2}\left[\frac{20}{3} + \frac{47}{3}\right]$$

$$\boxed{A = \frac{67}{6}}$$ — 1

Correct working out

Continue to next page.

Question 25

The random variable X has this probability distribution.

X	11	12	13	14	15
$P(X = x)$	0.3	0.2	0.1	0.3	0.1

a) Find the expected value. **1**

$\sum x P(x) = \boxed{12.7}$ — (1)

b) Find the standard deviation of x correct to 2 decimal places. **3**

* Variance $= \sum x^2 p(x) - (12.7)^2$ — (1)

$= 2.01$ — (1)

* $SD = \sqrt{Var} \approx 1.42$ — (1)

Or $SD = \sqrt{\sum x^2 P(x) - (12.7)^2}$

≈ 1.42

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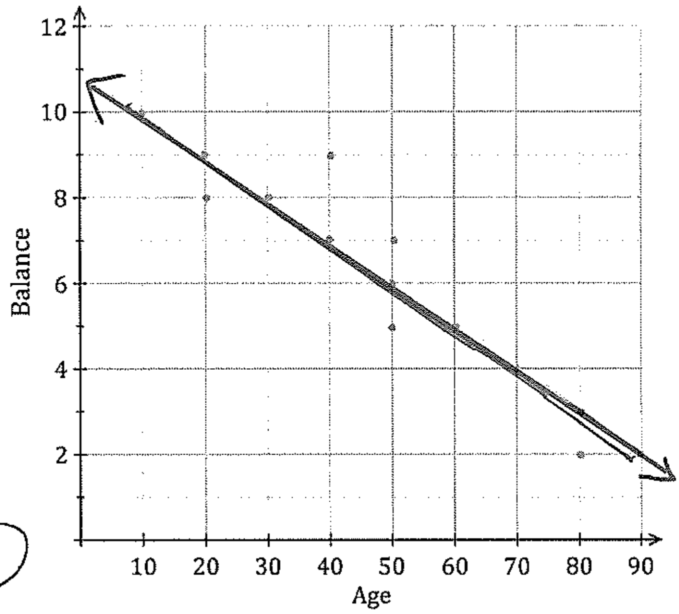
Question 26

The scatterplot below shows the relationship between age and balance.



1

a) Draw a line of best fit on the scatterplot **and** find the equation of this line. **3**

$m = -\frac{1}{10}$ $\boxed{y = -\frac{1}{10}x + 11}$ — 1

1

b) Hannah is 42 years old. Use your equation to find what is her expected balance? **1**

$y = \frac{-42}{10} + 11 = \boxed{6.8}$ — 1

c) Use the table of values below to find the value of the Pearson's correlation coefficient correct to two decimal places. **1**

Age	10	20	20	30	40	40	50	50	50	60	70	80	80	90
Balance	10	8	9	8	7	9	5	6	7	5	4	2	3	2

$\boxed{-0.96}$ — 1

Question 27

The first 3 terms of a sequence are:

$(x-1), (3x+2), (5x+5), ...$

a) Show that the sequence is arithmetic. **1**

$3x+2-(x-1) = 2x+3$

$5x+5-(3x+2) = 2x+3$

b) Find the sum of the first 50 terms of the sequence, in terms of x. **2**

$S_{50} = \frac{50}{2}\left[2(x-1)+49(2x+3)\right]$ — 1

$S_{50} = 25(100x+145)$

or $2500x + 3625$ — 1

Question 28 **2**

The table below gives some values of the probabilities of a standard normal distribution.

	first decimal place									
z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0.	0.5000	0.5398	0.5793	0.6179	0.6554	0.6915	0.7257	0.7580	0.7881	0.8159
1.	0.8413	0.8643	0.8849	0.9032	0.9192	0.9332	0.9452	0.9554	0.9641	0.9713
2.	0.9772	0.9821	0.9861	0.9893	0.9918	0.9938	0.9953	0.9965	0.9974	0.9981
3.	0.9987	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

Use the table above to find the probability $P(-1.5 \leq Z < 2.2)$

$= P(Z<2.2) - P(Z\leq -1.5)$ — 1

$= P(Z\leq 2.2) - P(Z\geq 1.5)$

$= 0.9861 - (1 - 0.9332)$

$= 0.9193$ — 1

Question 29

The particle's displacement is given by $x = (t^2 + 1)e^{-t}$ metres and velocity v m/s.

a) Find the initial displacement of the particle. **1**

$t=0$, $x=(0+1)\times 1$

$\boxed{x = 1\text{ m}}$

b) Express the velocity v in factorised form. **2**

$u = t^2+1 \qquad v = e^{-t}$

$u' = 2t \qquad v' = -e^{-t}$

$V = 2te^{-t} - e^{-t}(t^2+1)$ — 1

$V = e^{-t}(2t - t^2 - 1)$

$V = \boxed{-e^{-t}(t-1)^2}$ or $\boxed{\dfrac{-(t-1)^2}{e^t}}$ 1

c) Hence, find when the particle is at rest. **1**

$V=0 \Rightarrow (t-1)^2 = 0$

$\boxed{t = 1\text{ s}}$

Continue to next page.

Question 30 3

Solve $\ln(2\sin^2\theta - \cos\theta) = 0$ for $0 \le \theta \le 2\pi$.

$2\sin^2\theta - \cos\theta = 1$ — 1

$2(1-\cos^2\theta) - \cos\theta = 1$

$2 - 2\cos^2\theta - \cos\theta = 1$

$2\cos^2\theta + \cos\theta - 2 = -1$

$2\cos^2\theta + \cos\theta - 1 = 0$ — 1

$u = \cos\theta \quad 2u^2 + u - 1 = 0$

$2u^2 + 2u - u - 1 = 0$

$2u(u+1) - (u+1) = 0$

$(u+1)(2u-1) = 0$

$u = -1 \qquad u = \frac{1}{2}$

$\cos\theta = -1$ or $\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \pi$ — 1

Question 31 3

Evaluate

$$\int_0^{\frac{\pi}{4}} (\sin^2 x + \cos^2 x + \tan^2 x)\, dx$$

$= \int_0^{\frac{\pi}{4}} 1 + \tan^2 x$ — 1

$= \int_0^{\frac{\pi}{4}} \sec^2 x$ — 1

$= \left[\tan x\right]_0^{\frac{\pi}{4}}$

$= 1 - 0 = 1$ — 1

APPROXIMATELY THREE QUARTERS COMPLETE – 78 marks out of 100 complete

Question 32

The function $f(x) = \cos(3x) - 1$ is defined in the interval $0 \le x \le \frac{2\pi}{3}$.

a) What is the amplitude and period of the function? **2**

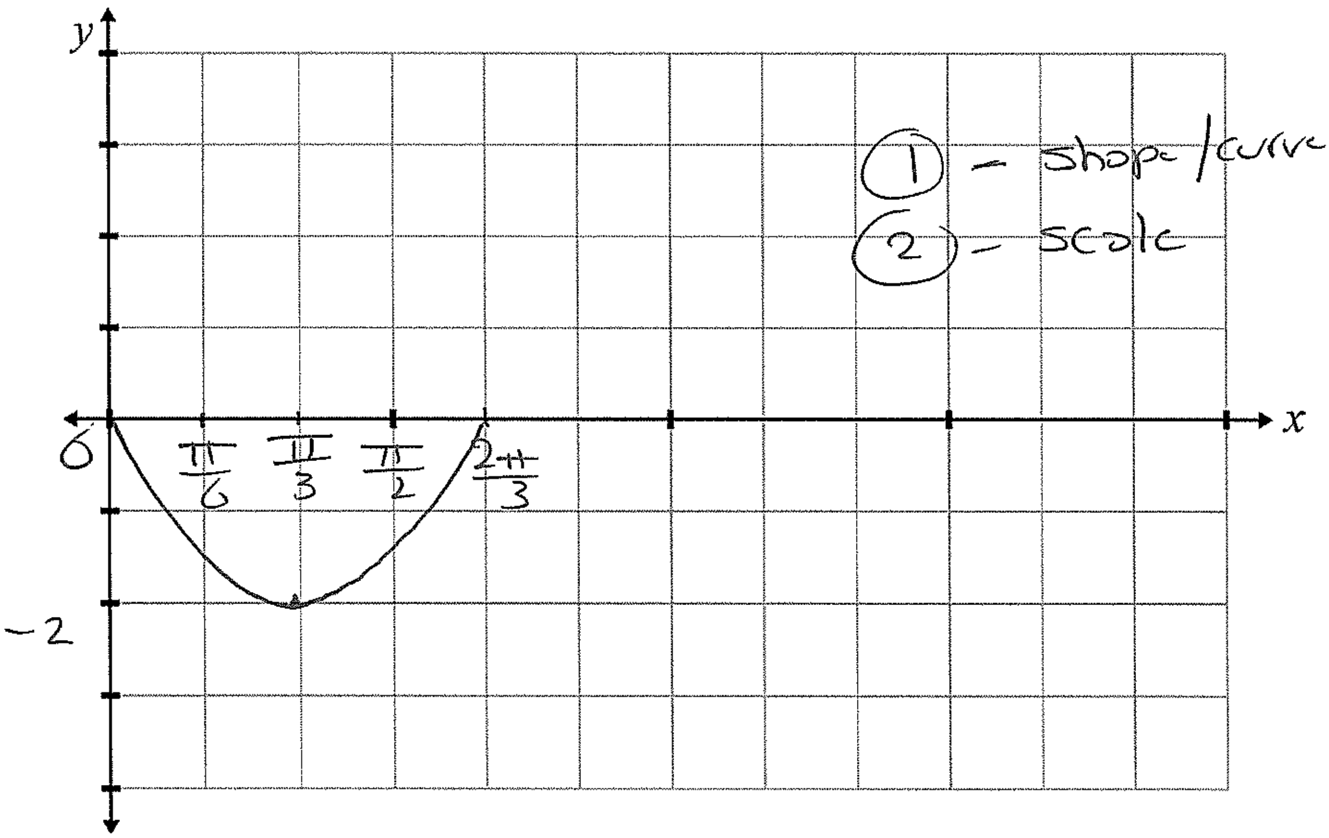
Amplitude = 1 — 1

Period = $\frac{2\pi}{3}$ — 1

b) What is the range of the function? **1**

$-2 \le y \le 0$ — 1

c) Sketch $f(x) = \cos(3x) - 1$ in the interval $0 \le x \le \frac{2\pi}{3}$. **2**



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Question 33

a) Show that **2**

$$\frac{d}{dx}(x\log_e(x) - x) = \log_e x$$

$x\left(\frac{1}{x}\right) + \log_e x - 1$ — 1

$= 1 + \log_e x - 1$

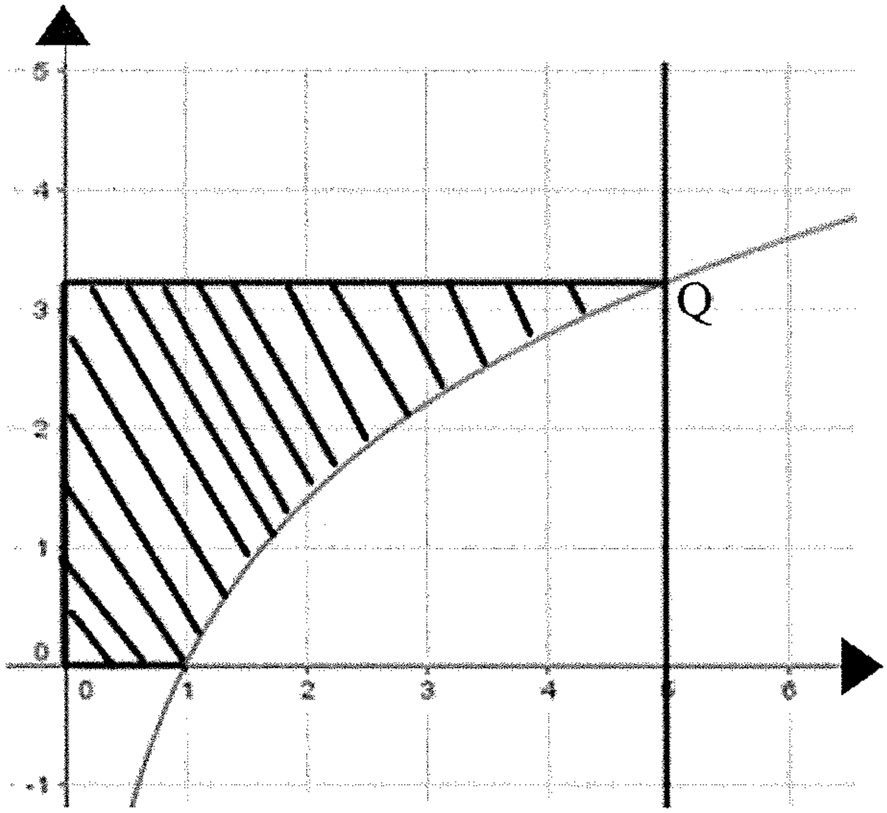
$= \log_e x$ — 1

b) Hence or otherwise find $\int 2\log_e(x)\,dx$ **1**

$2\int \log_e x = \boxed{2[x\log_e x - x]}$ — 1

Continue to next page.

a) The graph shows the curve $y = 2\log_e(x)$ which meets the line $x = 5$ at Q.



Using your answers from (i) and (ii), or otherwise, find the exact area of the shaded section.

$A = (2\log_e 5) \times 5 - \int_1^5 2\log_e x$ — 1

$= 10\ln 5 - 2\left[x\ln x - x\right]_1^5$

$= 10\ln 5 - 2[(5\ln 5 - 5) - (0 - 1)]$ — 1

$= 10\ln 5 - 2[5\ln 5 - 4]$

$= 10\ln 5 - 10\ln 5 + 8$

$= 8\ u^2$ — 1

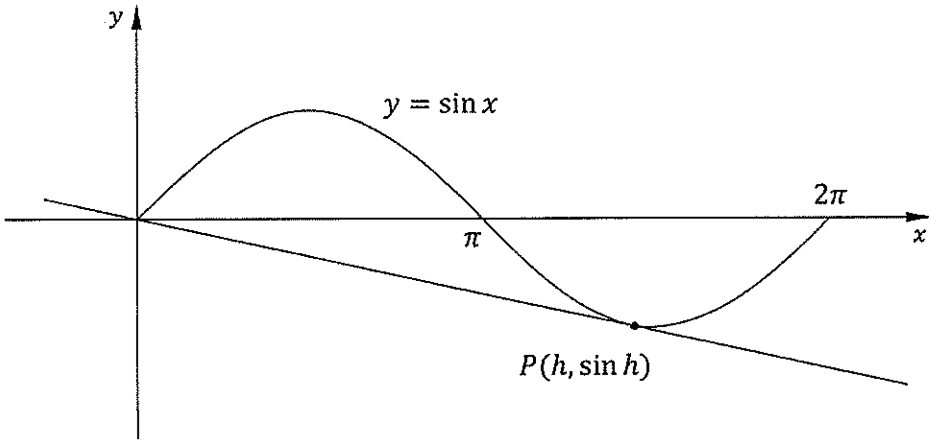
Question 34

3

The graph $y = \sin x$ only has one tangent in the domain $[\pi, 2\pi]$ that passes through the origin.

Let the point of contact of this tangent be $P(h, \sin h)$.

Prove that $h = \tan h$.



$$\left.\begin{aligned} y' &= \cos x \\ y'(\text{at } x=h) &= \cos h \\ m &= \cos h \end{aligned}\right\} - 1$$

$$\left.\begin{aligned} m &= \frac{\sin h - 0}{h - 0} \\ m &= \frac{\sin h}{h} \end{aligned}\right\} - 1$$

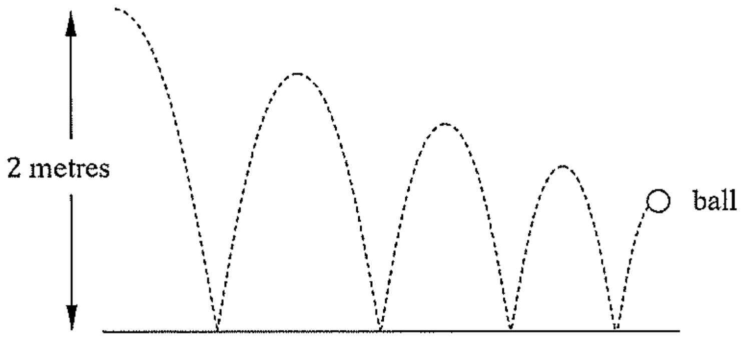
$$\Rightarrow \left.\begin{aligned} \frac{\sin h}{h} &= \cos h \\ \frac{\sin h}{\cos h} &= h \\ \boxed{\tan h = h} \end{aligned}\right\} - 1$$

Question 35

3

The Moon has a lower gravity than Earth, and there is no atmosphere to cause air resistance, so a ball would bounce higher and for much longer on the Moon than on Earth.

When a ball is dropped on the Moon each bounce is 95% as high as the previous bounce. When an identical ball is dropped on Earth each bounce is 50% as high as the previous bounce.



NOT TO SCALE

Two identical balls are dropped on the Moon and on Earth, each from a height of two metres. Calculate the difference in the total vertical distance travelled by these balls.

Moon: $a=2$, $r=0.95$

$$D = 2 \times \frac{2}{1-0.95} - 2 = 78 \quad 1$$

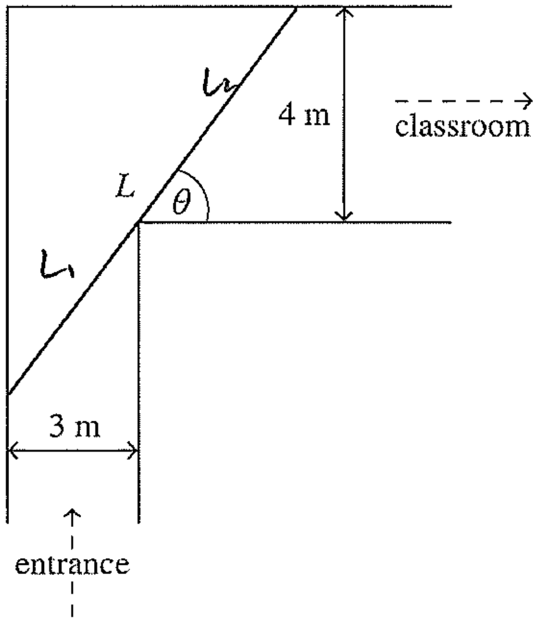
Earth: $a=2$, $r=0.5$

$$D = 2 \times \frac{2}{1-0.5} - 2 = 6 \quad 1$$

$$\text{Difference} = 78 - 6 = 72\text{ m} \quad 1$$

Question 36

A new whiteboard is being moved into a classroom. The whiteboard must be taken from the entrance, through the school's corridors and into the classroom. Two of the corridors are perpendicular to each other. The first corridor is 3 metres wide and the second corridor is 4 metres wide, as shown in the diagram. The length of the whiteboard is shown using L.



The whiteboard makes an angle θ to the horizontal on the corner of the corridors such that $0 < \theta < 90$.

a) Show that the length of the whiteboard, L, is **2**

$$\frac{3}{\cos\theta} + \frac{4}{\sin\theta}$$

$\cos\theta = \frac{3}{L_1}$ and $\sin\theta = \frac{4}{L_2}$ — 1

$\Rightarrow L_1 = \frac{3}{\cos\theta}$ and $L_2 = \frac{3}{\sin\theta}$

$\Rightarrow L = L_1 + L_2$

$= \frac{3}{\cos\theta} + \frac{3}{\sin\theta}$ — 1

Continue to next page.

b) In order to find the maximum possible length of the whiteboard such that it can be carried around the corner, you **must find the angle θ that minimises the function L** and then use that angle to **find that length of the whiteboard**. Hence, find that value of theta to the nearest minute and the length L to the nearest metre. **3**

- $L' = -3(-\sin\theta)\cos\theta^{-2} - 4\cos\theta\sin\theta^{-2}$

$L' = \frac{3\sin\theta}{\cos^2\theta} - \frac{4\cos\theta}{\sin^2\theta}$

$L' = \frac{3\sin^3\theta - 4\cos^3\theta}{\cos^2\theta\sin^2\theta}$ — 1

- $L' = 0$

$\Rightarrow 3\sin^3\theta - 4\cos^3\theta = 0$

$3\sin^3\theta = 4\cos^3\theta$

$\tan^3\theta = \frac{4}{3}$

$\tan\theta = \sqrt[3]{\frac{4}{3}} \Rightarrow \boxed{\theta \approx 47^\circ 45'}$ 1

- Min (Nature):

θ	47	θ	48
L'	-0.383	0	0.133
	\	—	/

- L (at $\theta = 47^\circ 45'$) $= \frac{3}{\cos\theta} + \frac{4}{\sin\theta}$

$\boxed{\approx 10\,m}$ — 1

END OF EXAM

Section II extra writing space

If you used this space, clearly indicate which question you are answering.

$$y = (x^2+2x)(x+1)^{-2}$$

$$y' = (x^2+2x)\left[-2(x+1)^{-3}(1)\right] + (x+1)^{-2}(2x+2)$$

$$= \frac{-2(x^2+2x)}{(x+1)^3} + \frac{2x+2}{(x+1)^2}$$

$$= \frac{-2x^2-4x+(2x+2)(x+1)}{(x+1)^3}$$

$$= \frac{\cancel{-2x^2}\cancel{-4x}+\cancel{2x^2}+\cancel{2x}+\cancel{2x}+2}{(x+1)^3}$$

$$= \frac{2}{(x+1)^3}$$

STUDENT NAME/NUMBER: ______________________

Section I

10 Marks

Attempt Questions 1-10.

Allow about 15 minutes for this section.

Select either A, B, C or D that best answers the question and indicate your choice with a cross (X) in the appropriate space on the grid below.

This page **must** be handed in with your answer booklet.

	A	B	C	D
1				X
2	X			
3	X			
4			X	
5		X		
6		X		
7			X	
8			X	
9	X			
10			X	